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Arakawa's Method Is a Finite-Element Method

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The nine-point second-order difference method of Arakawa for the two-dimensional stream function–vorticity equations of incompressible fluid flow comes from bilinear finite elements in rectangles. Furthermore, any nine-point second-order method obeying the conservation laws is a linear combination of two finite-element schemes, bilinear elements in rectangles and linear elements in triangles.

1. INTRODUCTION

This note is concerned with two-dimensional incompressible fluid flow. Throughout, boundaries will be ignored. If ζ denotes vorticity and ψ is the stream function, the equations of motion may be written as

$$\partial\zeta/\partial t = (\partial\zeta/\partial x)(\partial\psi/\partial y) - (\partial\zeta/\partial y)(\partial\psi/\partial x) \equiv J(\zeta, \psi), \tag{1}$$

$$\Delta\psi = \zeta. \tag{2}$$

The following conservation laws are satisfied by ζ and ψ.

$$\frac{d}{dt}\iint \zeta = 0 \qquad \text{(conservation of mean vorticity)}, \tag{3}$$

$$\frac{d}{dt}\iint \zeta^2 = 0 \qquad \text{(conservation of mean-square vorticity)}, \tag{4}$$

$$\frac{d}{dt}\iint \tfrac{1}{2}\,|\nabla\psi|^2 = 0 \qquad \text{(conservation of kinetic energy)}. \tag{5}$$

Numerical methods for solving (1)–(2) are subject to nonlinear instabilities, in particular aliasing error [3], unless the numerical method obeys the discrete analog of the conservation laws (3)–(5). The only well-known difference methods for (1)–(2) which obey all the conservation laws (3)–(5) are due to Arakawa [1]. On the other hand, semidiscrete finite-element approximations to (1)–(2) automatically satisfy the conservation laws, as shown in [2]. It is thus natural to ask if Arakawa's methods are related to finite-element methods. The object of this note is to announce the answer is yes, in the following sense. The nine-point second-order accurate difference scheme of Arakawa (Eq. (45) of [1]) is identical to that obtained by using bilinear finite elements in rectangles. Furthermore, any nine-point second-order scheme which obeys the conservation laws can be written as a linear combination of two finite-element schemes, one bilinear in rectangles, the other linear in triangles.

2. Semidiscrete Finite-Element Schemes

The crucial part of any difference method for (1.1)–(1.2) is the approximation to $J(\zeta, \psi)$. Let us consider the semidiscrete finite-element approximation to (1.1) [2]. With nodes $\{z_{ij} = (i\,\Delta x, j\,\Delta y)\}$ and basis functions $\{\phi_{ij}(x, y)\}$ we write

$$\zeta^h(x, y, t) = \sum \zeta_{ij}(t)\, \phi_{ij}(x, y),$$
$$\psi^h(x, y, t) = \sum \psi_{ij}(t)\, \phi_{ij}(x, y).$$

In what follows the basis functions will be standard "hill" functions: $\phi_{ij}(z_{mn}) = 1$ if $i = m$ and $j = n$, otherwise $\phi_{ij}(z_{mn}) = 0$ (see [4] or [5]). With this choice of basis, $\zeta_{ij}(t) = \zeta^h(i\,\Delta x, j\,\Delta y, t)$. We then convert (1.1) to the weak or Galerkin form

$$\iint \frac{\partial \zeta^h}{\partial t}\, \phi_{ij} = \iint J(\zeta^h, \psi^h)\, \phi_{ij}\,, \qquad \text{all } (i, j),$$

and obtain a system of ordinary differential equations

$$M\dot{\boldsymbol{\zeta}}^h = K(\psi)\, \boldsymbol{\zeta}^h,$$

where

$$\boldsymbol{\zeta}^h = \boldsymbol{\zeta}^h(t) = (\ldots, \zeta_{ij}(t), \ldots)^T, \qquad M_{(i,j)(i',j')} = \iint \phi_{ij}\phi_{i'j'}\,,$$

and

$$K(\psi)_{(i,j)(i',j')} = \iint J(\phi_{i'j'}\,, \psi^h)\, \phi_{ij} = \sum_{i'',j''} \psi_{i''j''}(t) \iint J(\phi_{i'j'}\,, \phi_{i''j''})\, \phi_{ij}\,.$$

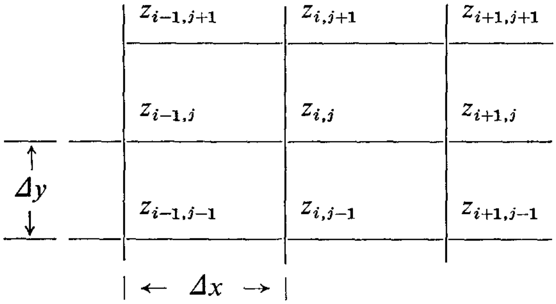
The generic equation of the above system is

$$\sum_{i',j'} M_{(i,j)(i',j')} \dot{\zeta}_{i'j'} = \sum_{i',j'} K(\psi)_{(i,j)(i',j')} \zeta_{i'j'}\,. \tag{1}$$

The term on the left is equal to $\Delta x\, \Delta y(\dot{\zeta}_{ij} + O(\Delta x^2) + O(\Delta y^2))$. (Note that some loss of accuracy will result if M is replaced by I, the identity matrix; see [6]). The right-hand side is more interesting, as it is the approximation to $\Delta x\, \Delta y\, J(\zeta, \psi)$. The coefficients are

$$K_{(i,j)(i',j')} = \iint \left\{ \frac{\partial \phi_{i'j'}}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \phi_{i'j'}}{\partial y} \frac{\partial \psi}{\partial x} \right\} \phi_{ij}\,.$$

Suppose we use a uniform rectangular grid and index the nodes as follows:



If we use bilinear elements, those of the form $a + bx + cy + dxy$ in each rectangle, the term $K_{(i,j)(i',j')}$ is zero unless $z_{i'j'}$ is a neighbor of z_{ij}, i.e., $|\, i - i' \,| \leqslant 1$ and $|\, j - j' \,| \leqslant 1$. There are essentially two cases: connections with corners $(i \pm 1, j \pm 1)$, and horizontal or vertical connections $(i, j \pm 1)$ and $(i \pm 1, j)$. The pattern for the former is established by a straightforward computation which yields

$$K_{(i,j)(i+1,j+1)} = (\psi_{i,j+1} - \psi_{i+1,j})/12.$$

We represent this by Fig. 1, the other corner connections are represented in Fig. 2. For example, $K_{(i,j)(i-1,j+1)} = (\psi_{i-1,j} - \psi_{i,j+1})/12$.

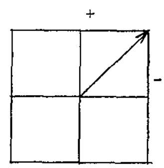


FIGURE 1

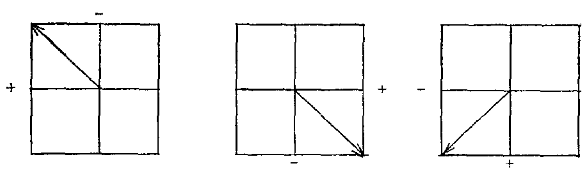


FIGURE 2

Another straightforward computation establishes that

$$K_{(i,j)(i+1,j)} = (\psi_{i,j+1} - \psi_{i,j-1} + \psi_{i+1,j+1} - \psi_{i+1,j-1})/12,$$

which can be represented by Fig. 3. The other connections are represented in Fig. 4. Finally, the diagonal term $K_{(i,j)(i,j)}$ is zero.

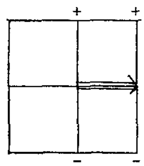


FIGURE 3

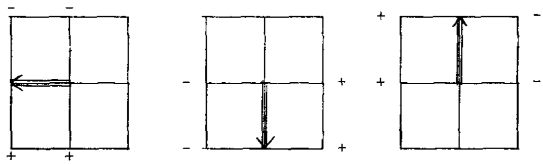


FIGURE 4

To be explicit, Eq. (1) becomes

$$\begin{aligned}
&(1/36)\{16\dot{\zeta}_{ij} + 4(\dot{\zeta}_{i+1,j} + \dot{\zeta}_{i,j+1} + \dot{\zeta}_{i-1,j} + \dot{\zeta}_{i,j-1}) \\
&\quad + \dot{\zeta}_{i+1,j+1} + \dot{\zeta}_{i+1,j-1} + \dot{\zeta}_{i-1,j+1} + \dot{\zeta}_{i-1,j-1}\} \\
&\quad = (1/12\,\Delta x\,\Delta y)\{(\psi_{i,j+1} - \psi_{i+1,j})\,\zeta_{i+1,j+1} + (\psi_{i-1,j} - \psi_{i,j+1})\,\zeta_{i-1,j+1} \\
&\quad + (\psi_{i+1,j} - \psi_{i,j-1})\,\zeta_{i+1,j-1} + (\psi_{i,j-1} - \psi_{i-1,j})\,\zeta_{i-1,j-1} \\
&\quad + (\psi_{i,j+1} - \psi_{i,j-1} + \psi_{i+1,j+1} - \psi_{i+1,j-1})\,\zeta_{i+1,j} \\
&\quad + (\psi_{i,j-1} - \psi_{i,j+1} + \psi_{i-1,j-1} - \psi_{i-1,j+1})\,\zeta_{i-1,j} \\
&\quad + (\psi_{i+1,j} - \psi_{i-1,j} + \psi_{i+1,j-1} - \psi_{i-1,j-1})\,\zeta_{i,j-1} \\
&\quad + (\psi_{i-1,j} - \psi_{i+1,j} + \psi_{i-1,j+1} - \psi_{i+1,j+1})\,\zeta_{i,j+1}\}.
\end{aligned} \tag{2}$$

The right-hand side of this equation is identical to Arakawa's first second-order approximation, though his derivation was altogether different.

If instead of a rectangular grid we use a triangular grid (see Fig. 5), we can then

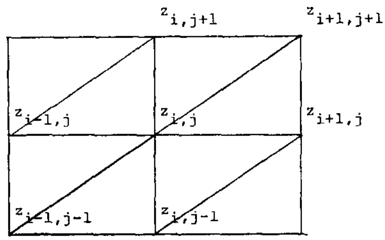


FIGURE 5

use linear elements, those of the form $a + bx + cy$ in each triangle. In this case, (1) becomes

$$
\begin{aligned}
(1/12)\{6\dot{\zeta}_{ij} &+ \dot{\zeta}_{i,j-1} + \dot{\zeta}_{i-1,j-1} + \dot{\zeta}_{i-1,j} + \dot{\zeta}_{i+1,j+1} + \dot{\zeta}_{i+1,j}\} \\
&= (1/6\,\Delta x\,\Delta y)\{(\psi_{i+1,j+1} - \psi_{i,j-1})\,\zeta_{i+1,j} + (\psi_{i-1,j} - \psi_{i+1,j+1})\,\zeta_{i,j+1} \\
&\quad + (\psi_{i-1,j-1} - \psi_{i,j+1})\,\zeta_{i,j-1} + (\psi_{i+1,j} - \psi_{i-1,j-1})\,\zeta_{i-1,j} \\
&\quad + (\psi_{i,j+1} - \psi_{i+1,j})\,\zeta_{i+1,j+1} + (\psi_{i,j-1} - \psi_{i-1,j})\,\zeta_{i-1,j-1}\}.
\end{aligned} \tag{3}
$$

The right-hand side of this equation is a conservative seven-point second-order approximation to $J(\zeta, \psi)$ which seems to be new. It may be computationally more attractive than (2) as fewer multiplications are involved.

3. NINE-POINT CONSERVATIVE SCHEMES

The object of this section is to extend the analysis of Arakawa [1] and derive all possible conservative nine-point approximations to the jacobian $J(\zeta, \psi)$. Begin by considering, as in [1], a general finite-difference approximation which is convenient to write as (where for simplicity we take $\Delta x = \Delta y = h$)

$$
\begin{aligned}
J(\zeta, \psi)_{ij} &= \frac{1}{12h^2} \sum_{\alpha,\beta} a_{\alpha,\beta}(i,j)\,\zeta_{i+\alpha,j+\beta}\,, \\
&= \frac{1}{12h^2} \sum_{\gamma,\delta} b_{\gamma,\delta}(i,j)\,\psi_{i+\gamma,j+\delta}\,, \\
&= \frac{1}{12h^2} \sum_{\substack{\alpha,\beta \\ \gamma,\delta}} c_{\gamma\delta}^{\alpha\beta}\zeta_{i+\alpha,j+\beta}\psi_{i+\gamma,j+\delta}\,.
\end{aligned} \tag{1}
$$

The crucial point is that $c_{\gamma\delta}^{\alpha\beta}$ is independent of i and j.

In [1] the following conditions for such a scheme to obey the conservation laws are derived (see Eqs. (18), (20), (34), and (35) of [1]).

$$\sum_{\alpha,\beta} a_{\alpha\beta}(i,j) = 0 = \sum_{\gamma,\delta} b_{\gamma\delta}(i,j), \tag{2}$$

$$c_{\gamma\delta}^{\alpha\beta} = -c_{\gamma-\alpha,\delta-\beta}^{-\alpha,-\beta} = -c_{-\gamma,-\delta}^{\alpha-\gamma,\beta-\delta}, \qquad \text{for all } \alpha,\ \beta,\ \gamma,\ \delta. \tag{3}$$

From (3) we get the following chain of equalities.

$$c_{\gamma\delta}^{\alpha\beta} = -c_{\gamma-\alpha,\delta-\beta}^{-\alpha,-\beta} = c_{\alpha-\gamma,\beta-\delta}^{-\gamma,-\delta} = -c_{\alpha\beta}^{\gamma\delta} = c_{-\alpha,-\beta}^{\gamma-\alpha,\delta-\beta} = -c_{-\gamma,-\delta}^{\alpha-\gamma,\beta-\delta}. \tag{4}$$

Consider now only nine-point schemes; thus α, β, γ, δ take on the values $-1, 0, 1$, so there are 81 unknowns $c_{\gamma\delta}^{\alpha\beta}$. The constraints (3)–(4) force many of these unknowns to be zero. For example, if $\gamma = -1$ and $\alpha = 1$, or if $\gamma = 1$ and $\alpha = -1$, then (4) plus the restriction to nine-point schemes force $c_{\gamma\delta}^{\alpha\beta} = 0$, and similarly for β and δ. Also, if $\alpha = \beta = 0$, or if $\gamma = \delta = 0$, then (4) implies $c_{\gamma\delta}^{\alpha\beta} = 0$. Again, if $\alpha = \gamma$ and $\beta = \delta$, then (4) plus the previous remark imply $c_{\gamma\delta}^{\alpha\beta} = -c_{\gamma-\alpha,\delta-\beta}^{-\alpha,-\beta} = 0$. In this way 57 of the 81 unknowns $c_{\gamma\delta}^{\alpha\beta}$ turn out to be zero. The remaining 24 divide into four groups of six each under (4), leaving only four unknowns free. Perhaps a diagram would make things clearer (see Fig. 6). The four free unknowns have been taken to be c_{11}^{01}, c_{11}^{10}, c_{10}^{01}, and $c_{10}^{1,-1}$.

δ	β \ α (γ)	-1 (γ=-1)	0 (γ=-1)	1 (γ=-1)	-1 (γ=0)	0 (γ=0)	1 (γ=0)	-1 (γ=1)	0 (γ=1)	1 (γ=1)
-1	-1	0	$-c_{11}^{10}$	0	c_{11}^{10}	0	c_{10}^{01}	0	$-c_{10}^{01}$	0
	0	$-c_{11}^{01}$	0	0	$c_{10}^{1,-1}$	0	c_{11}^{01}	0	0	$-c_{10}^{1,-1}$
	1	0	0	0	0	0	0	0	0	0
0	-1	c_{11}^{01}	$-c_{10}^{1,-1}$	0	0	0	0	0	$-c_{11}^{01}$	$c_{10}^{1,-1}$
	0	0	0	0	0	0	0	0	0	0
	1	$-c_{10}^{01}$	c_{11}^{10}	0	0	0	0	0	c_{10}^{01}	$-c_{11}^{10}$
1	-1	0	0	0	0	0	0	0	0	0
	0	c_{10}^{01}	0	0	$-c_{11}^{10}$	0	$-c_{10}^{01}$	0	0	c_{11}^{10}
	1	0	$c_{10}^{1,-1}$	0	$-c_{10}^{1,-1}$	0	$-c_{11}^{01}$	0	c_{11}^{01}	0

FIGURE 6

Equation (2) implies $\sum_{\alpha,\beta;\gamma,\delta} c_{\gamma\delta}^{\alpha\beta}\psi_{i+\gamma,j+\delta} = 0$ and hence, since ψ is arbitrary, $\sum_{\alpha,\beta} c_{\gamma\delta}^{\alpha\beta} = 0$ for each pair (γ, δ). This places further constraints on the $c_{\gamma\delta}^{\alpha\beta}$; we see $c_{11}^{10} = -c_{11}^{01}$ and $c_{10}^{1,-1} = -c_{10}^{01}$.

The condition that (1) be a second-order approximation to $J(\zeta, \psi)_{ij}$ has yet to be imposed. Setting (1) equal to

$$((\partial\zeta/\partial x)(\partial\psi/\partial y) - (\partial\zeta/\partial y)(\partial\psi/\partial x))_{ij} + O(h^2)$$

yields many additional equations which the $c_{\gamma\delta}^{\alpha\beta}$ must satisfy. If we expand (1) in a formal Taylor series we get

$$\frac{1}{12h^2} \sum_{\substack{\alpha\beta \\ \gamma\delta}} c_{\gamma\delta}^{\alpha\beta} \left\{ \sum_{m,n} \alpha^m \beta^n \frac{h^{m+n}}{m!\, n!} \frac{\partial^{m+n}\zeta}{\partial x^m\, \partial y^n} \right\} \left\{ \sum_{p,q} \gamma^p \delta^q \frac{h^{p+q}}{p!\, q!} \frac{\partial^{p+q}\psi}{\partial x^p\, \partial y^q} \right\}_{ij}$$

$$= ((\partial\zeta/\partial x)(\partial\psi/\partial y) - (\partial\zeta/\partial y)(\partial\psi/\partial x))_{ij} + O(h^2),$$

and hence,

$$\frac{1}{12} \sum_{\substack{\alpha\beta \\ \gamma\delta}} \alpha\delta c_{\gamma\delta}^{\alpha\beta} = 1, \tag{5}$$

$$\frac{1}{12} \sum_{\substack{\alpha\beta \\ \gamma\delta}} \beta\gamma c_{\gamma\delta}^{\alpha\beta} = -1, \tag{6}$$

$$\sum_{\substack{\alpha\beta \\ \gamma\delta}} \alpha^m \beta^n \gamma^p \delta^q c_{\gamma\delta}^{\alpha\beta} = 0,$$

$$\text{for all other } m, n, p, q \text{ such that } \quad m + n + p + q \leqslant 3. \tag{7}$$

Equation (6) follows from (5), since $\sum \alpha\delta c_{\gamma\delta}^{\alpha\beta} = -\sum \alpha\delta c_{\alpha\beta}^{\gamma\delta} = -\sum \beta\gamma c_{\gamma\delta}^{\alpha\beta}$. Equation (5) in conjunction with the previous work yields $c_{11}^{10} - c_{10}^{01} = 2$. The conditions (2)–(4) ensure that (7) is satisfied. Thus there is a one-parameter family of possible difference schemes. The choice $c_{10}^{01} = -1$ yields the rectangular bilinear scheme (call it J_R) of Section 2, while the choice $c_{10}^{01} = 0$ gives the triangular linear scheme (call it J_T) of Section 2, and any other nine-point conservative difference method is a linear combination ($\lambda J_R + (1 - \lambda) J_T$ for any real number λ) of these two methods. Notice further that the finite element $\lambda\phi_R + (1 - \lambda)\phi_T$ gives rise to the jacobian $\lambda J_R + (1 - \lambda) J_T$.

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REFERENCES

1. A. ARAKAWA, *J. Computational Phys.* **1** (1960), 119–143.
2. G. FIX, *SIAM J. Appl. Math.* **28**, to appear.
3. P. ROACHE, "Computational Fluid Dynamics," Hermosa, Albuquerque, NM, 1972.
4. M. SCHULTZ, "Spline Analysis," Prentice-Hall, Englewood Cliffs, NJ, 1973.
5. G. STRANG AND G. FIX, "An Analysis of the Finite Element Method," Prentice-Hall, Englewood Cliffs, NJ, 1973.
6. B. WENDROFF, Spline-Galerkin Methods for Initial-Value Problems with Variable Coefficients, in "Conference on the Numerical Solution of Differential Equations," (G. A. Watson, Ed.) Springer-Verlag Lecture Notes in Mathematics No. 363, Springer-Verlag, New York, 1974.